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Multiplicity Results for a Class of Semilinear Elliptic and Parabolic Boundary Value Problems

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INTRODUCTION

Let Ω be an open bounded domain in R^N $(N \geqslant 1)$ with smooth boundary $\partial\Omega$ of class $C^{2+\gamma}$ $(\gamma > 0)$. Let A be a second-order, selfadjoint, strongly elliptic operator with smooth real-valued coefficients defined in Ω such that $-Au \geqslant 0$ in Ω and $u \mid \partial\Omega = 0$ imply $u \geqslant 0$ on Ω—for example, $A = \Delta$. Let $\lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ denote the eigenvalues of $-A$ with Dirichlet boundary conditions, each counted as often as its multiplicity and let $g: R \to R$ be of class C^1 and $p \in C^\gamma(\bar{\Omega})$. Under the assumptions that $\lim_{u \to -\infty} (g(u)/u) \equiv \alpha$, $\lim_{u \to +\infty} (g(u)/u) = \beta$ exist and $\alpha < \lambda_1 < \beta$, the boundary value problem

$$Au + g(u) = p(x), \qquad x \in \Omega,\ u \mid \partial\Omega = 0 \tag{P}$$

has been widely investigated during the past decade beginning with a result due Ambrosetti and Prodi [2]. Before Ambrosetti and Prodi, Hammerstein, in a classical paper on nonlinear integral equations [7], proved a result which implies that the problem is solvable provided that $g(u)/u \leqslant \gamma < \lambda_1$ for $|u|$ sufficiently large. Ambrosetti and Prodi assumed that $f''(u) > 0$ for all u, $0 < \alpha < \lambda_1$, and $\lambda_1 < \beta < \lambda_2$. Under these assumptions they showed

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that $C^\gamma(\bar{\Omega}) = E_0 \cup E_1 \cup E_2$ where E_0 and E_2 are disjoint open sets, $E_1 = \partial E_0 = \partial E_2$, and the problem has exactly k solutions for $p \in E_k$, $k = 0, 1, 2$. The structure of these sets was elucidated by Berger and Podolak [3]. To describe their result, here and in the remainder of this section we let ϕ_1 denote the normalized positive eigenfunction corresponding to λ_1 and h a smooth function which satisfies $\langle h, \phi_1 \rangle = 0$ where $\langle\ ,\ \rangle$ is the $L^2(\Omega)$ inner product. Under the same conditions on g as assumed by Ambrosetti and Prodi, Berger and Podolak showed that there exists a number $s_0 = s_0(h)$ such that $h + s\phi_1 \in E_0$ if $s < s_0$, $h + s\phi_1 \in E_1$ if $s = s_0$, and $h + s\phi_1 \in E_2$ if $s > s_0$. Independently of Berger and Podolak, Kazdan and Warner gave a near characterization of the range of a much wider class of semilinear elliptic operators. If it is only assumed that g is a smooth function such that $\lim_{u \to -\infty} \sup g(u)/u < \lambda_1 < \lim_{u \to \infty} \inf g(u)/u$ the results of [9] show that if $p = h + s\phi_1$, then there exists s_0 such that (P) is solvable if $s > s_0$ but not if $s < s_0$. Simultaneously, Dancer [4] and Amann and Hess [1] showed that if g satisfies the above conditions and in addition $g(u)/u$ is bounded on $[1, \infty)$, then (P) has at least two solutions if $p = h + s\phi_1$ and $s > s_0$ and at least one solution if $s = s_0$. Actually, Dancer only requires that $g(u)/u$ grow no faster than u^r as $u \to +\infty$ where $r > 1$ depends on $\dim \Omega$. In [10], the present authors showed that if $\lim_{u \to -\infty} \sup g(u)/u < \lambda_1$ and $\lambda_{2n} < \lim_{u \to \infty} g(u)/u < \lambda_{2n+1}$ $(n \geqslant 1)$, then there exists $s_1 \geqslant s_0$ such that (P) has at least three solutions if $p = h + s\phi_1$ and $s \geqslant s_1$. We pointed out that generically there should be four solutions under these hypotheses and we further conjectured that if $\lambda_k < \lim_{u \to \infty} g(u)/u < \lambda_{k+1}$, there are at least $2k$ solutions for s large. In [11] we proved this conjecture in the one-dimensional case.

Recently, Hofer [8] gave further support to the conjecture by showing that if $\lim_{u \to -\infty} \sup g(u)/u < \lambda_1$, $\lambda_k < \lim_{u \to \infty} g(u)/u < \lambda_{k+1}$, where $k \geqslant 2$ and $h = h_1 + s\phi_1$ then (P) has at least four solutions when $p = h + s\phi_1$ for s large. Previously, Solimini [15] had extended the result of [10] by showing the existence of at least three solutions for s sufficiently large assuming the same conditions on g as Hofer.

Hofer studied (P) using variational methods. In this paper we obtain multiplicity results using a reduction method and degree theory. We assume the existence of $\lim_{u \to -\infty} g(u)/u = \alpha$ and $\lim_{u \to \infty} g(u)/u = \beta$. Under the assumptions that λ_2 is simple, $\alpha < \lambda_1$, and $\lambda_2 < \beta \leqslant \lambda_3$ we show that for $p = h + s\phi_1$, (P) has at least four solutions for sufficiently large s. Hofer's methods do not appear to be applicable for the case $\beta = \lambda_3$. We also show that if λ_3 has odd multiplicity, there exists $\bar{\beta} > \lambda_3$ such that if $\alpha < \lambda_1$, $\lambda_3 < \beta \leqslant \bar{\beta}$, and $p = h + s\phi_1$, then (P) has at least *five* solutions for s sufficiently large. This appears to be the first general condition which implies the existence of five solutions for this type of problem when $\dim \Omega > 1$. Moreover, the result does not seem obtainable with Hofer's methods. Our method also shows that the solutions persist under perturbations of the linear part of the differential equation by a nonselfadjoint operator.

We apply the methods used to study the elliptic problem to the parabolic problem

$$\begin{aligned} u_{xx}-u_t+g(u)&=s\phi_1(x)+h(x,t), && 0<x<\pi,\ -\infty<t<\infty \\ u(t,0)&=u(t,\pi)=0, && -\infty<t<\infty \\ u(t+T,x)&\equiv u(t,x), && T>0,\ -\infty<t<\infty,\ 0\leqslant x\leqslant\pi. \end{aligned} \tag{P1}$$

Here, it is assumed that g is continuous, $h(x, t+T)\equiv h(x,t)$, h is locally square-integrable on the strip $-\infty<t<\infty$, $0<x<\pi$, $\phi_1(x)=\sqrt{2/\pi T}\sin x$, and h and ϕ_1 are orthogonal on $[0,T]\times[0,\pi]$. We show that if $T<2\pi/\sqrt{15}$, there exists a number $\bar{\beta}>4$ such that if $0<\lim_{u\to-\infty} g(u)/u<1$ and $4<\lim_{u\to\infty} g(u)/u<\bar{\beta}$, then for s large (P1) has at least four weak solutions which are continuous and T-periodic for $-\infty<t<\infty$, $0\leqslant x\leqslant\pi$, which satisfy the boundary conditions at $x=0$ and $x=\pi$, and which have distributional derivatives u_t, u_x, and u_{xx} which are locally square-integrable.

Since the linear part of the differential equation in (P1) is nonselfadjoint, variational methods appear to be completely inapplicable to this problem.

In the last part of this paper we apply the methods that we have developed to the piecewise linear problem

$$\Delta u+\beta u^+-\alpha u^-=\phi_1 \text{ in } \Omega, \qquad u\mid\partial\Omega=0 \tag{P2}$$

where Ω is the square $0<x<\pi$, $0<y<\pi$ and $\phi_1(x,y)=\sin x\sin y$. We show that there are at least six solutions if $\alpha<\lambda_1$, $\lambda_2=\lambda_3<\beta<\lambda_4$, and at least eight solutions if $\alpha<\lambda_1$ and $\lambda_4<\beta<\lambda_5$.

2. A Piecewise Linear Problem

In this section we study the nonlinear Dirichlet problem

$$Au+f(u)=\phi_1 \text{ in } \Omega \qquad u\mid\partial\Omega=0 \tag{1}$$

where A and Ω are as in the introductory section, ϕ_1 is an eigenfunction corresponding to λ_1 satisfying $(\phi_1,\phi_1)_0=1$, where $(\ ,\ )_0$ is the usual $L^2(\Omega)$ inner product, and f is the piecewise linear function defined by

$$f(s)=\beta s^+-\alpha s^-. \tag{2}$$

We assume that

$$\alpha<\lambda_1, \qquad \lambda_2<\beta<\lambda_3 \tag{3}$$

which implies that λ_2 is simple. We shall show that these hypotheses imply the existence of at least four solutions of (1) which are stable under perturbations of the right-hand side, by nonlinear perturbations of f, and linear perturbations of the selfadjoint operator A which need not be selfadjoint.

It is well known that ϕ_1 does not vanish in Ω and $\partial\phi_1/\partial n$ does not vanish on $\partial\Omega$, where $\partial\phi_1/\partial n$ denotes the directional derivative of ϕ_1 in the direction of the outer unit normal on $\partial\Omega$. We may assume that

$$\phi_1(x) > 0 \text{ in } \Omega, \qquad \frac{\partial\phi_1}{\partial n} < 0 \text{ on } \partial\Omega. \tag{4}$$

We let $\phi_2(x)$ denote an eigenfunction corresponding to λ_2 such that $(\phi_2, \phi_2)_0 = 1$ and we let V denote the two-dimensional subspace of $L^2(\Omega)$ spanned by ϕ_1 and ϕ_2. The orthogonal projection of $L^2(\Omega)$ onto V will be denoted by P; i.e. $Pu = (\phi_1, u)_0 \phi_1 + (\phi_2, u)_0 \phi_2$ for $u \in L^2(\Omega)$. If we let W denote the set of $w \in L^2(\Omega)$ such that $(w, v)_0 = 0$ for all $v \in V$, then W is the range of $I - P$ where I is the identity on $L^2(\Omega)$ and $L^2(\Omega) = V \oplus W$.

LEMMA 2.1. *Given $v \in V$ there exists a unique $\theta(v) \in W \cap C^{2+\gamma}(\bar{\Omega})$ such that*

$$A\theta(v) + (I-P)f(v + \theta(v)) = 0, \qquad \theta(v) \mid \partial\Omega = 0, \tag{5}$$

and the mapping θ: $V \to W$ is Lipshitz continuous.

Proof. Choose $b \geqslant 0$ so that $\alpha + b > 0$ and let $f_1(s) = f(s) + bs$. The assertion of the lemma is equivalent to the existence of a unique $w \in W \cap C^{2+\gamma}(\bar{\Omega})$ such that $(A - b)\, w + (I - P) f_1(v + w) = 0$ and $w \mid \partial\Omega = 0$. Now f_1 maps $L^2(\Omega)$ continuously into itself and $(A - b)^{-1}$, which is a one-to-one continuous map from $C^{\gamma}(\bar{\Omega})$ onto the subspace of functions in $C^{2+\gamma}(\bar{\Omega})$ which vanish on $\partial\Omega$, extends to a continuous mapping from $L^2(\Omega)$ onto the Sobolev space $H_2(\Omega) \cap H_1^0(\Omega) \subset L^2(\Omega)$.

Let $\| \; \|$ denote the $L^2(\Omega)$ norm. If h is a smooth function, $(A - b)\, u = (I - P)\, h$ in Ω, and $u \mid \partial\Omega = 0$, then $-(\lambda_k + b)(u, \phi_k)_0 = (Au - bu, \phi_k)_0 = ((I - P)\, h, \phi_k)_0 = 0$ for $k = 1, 2$. Hence, $(u, \phi_k)_0 = 0$ for $k = 1, 2$. From the variational characterization of λ_3 we have $(\lambda_3 + b)\|u\|^2 \leqslant (-Au + bu, u)_0 = -((I - P)\, h, u)_0 \leqslant \|(I - P)\, h\| \; \|u\| \leqslant \|h\| \; \|u\|$, so $\|(A - b)^{-1}\, h\| = \|u\| \leqslant (\lambda_3 + b)^{-1} \; \|h\|$. Since $C^{\gamma}(\bar{\Omega})$ is dense in $L^2(\Omega)$, if we consider $(A - b)^{-1}$ as a mapping from $L^2(\Omega)$ into itself, then $\|(A - b)^{-1}(I - P)\| \leqslant (\lambda_3 + b)^{-1}$.

The Nemytskii map associated with f_1 maps $L^2(\Omega)$ continuously into itself and $(A - b)^{-1}\,(I - P) = (I - P)(A - b)^{-1}$. Consequently, $w \to -(A - b)^{-1}\,(I - P) f_1\,(v + w)$ defines a continuous mapping of W into itself. Moreover, since $|f_1(s_1) - f_1(s_2)| \leqslant (\beta + b)\, |s_1 - s_2|$ for arbitrary s_1 and

s_2, it follows that for arbitrary w_1 and w_2 in W that $\|(A-b)^{-1}(I-P)[f_1(v+w_1)-f_1(v+w_2)]\| \leqslant \delta \|w_1-w_2\|$ where $\delta = (\beta+b)(\lambda_3+b)^{-1} < 1$. The contraction mapping principle implies the existence of a unique $w^* \in W$ such that $w^* = -(A-b)^{-1}(I-P)f_1(v+w^*)$. It follows that $w^* \in \mathring{H}_1^0(\Omega)$ and

$$\begin{aligned}(A-b)\,w^* &= -(I-P)[f(v+w^*)+b(v+w^*)]\\ &= -(I-P)f(v+w^*)-bw^*\end{aligned}$$

in the weak sense. Since functions in V are smooth and f grows linearly at both $-\infty$ and $+\infty$, it follows from standard bootstrap arguments that $w^* \in C^{2+\gamma}(\bar{\Omega})$, $Aw^* + (I-P)f(v+w^*)=0$ in Ω, and $w^* \mid \partial\Omega = 0$. Setting $w^* = \theta(v)$, this proves the first part of the lemma. If $w_1^* = \theta(v_1)$, $w_2^* = \theta(v_2)$, where $v_1, v_2 \in V$, then from the above we have

$$\begin{aligned}w_1^* - w_2^* = &-(A-b)(I-P)[f_1(v_1+w_1^*)-f_1(v_1+w_2^*)]\\ &+(A-b)(I-P)[f_1(v_2+w_2^*)-f_1(v_1+w_2^*)].\end{aligned}$$

Therefore, if δ is as above, then

$$(1-\delta)\,\|w_1^*-w_2^*\| \leqslant \delta\,\|v_1-v_2\|.$$

This shows that the mapping $\theta\colon V \to W$ is Lipshitz and the lemma is proved.

Suppose that u is a solution of (1) and let $u = v+w$ with $v \in V$, $w \in W$. Since $AP = PA$ and $v \mid \partial\Omega = 0$, it follows that

$$Aw + (I-P)f(v+w) = 0, \qquad w \mid \partial\Omega = 0 \tag{6}$$

and

$$Av + Pf(v+w) = \phi_1. \tag{7}$$

From (6) and Lemma (1) we conclude that $w = \theta(v)$ and therefore, from (7), we have

$$Av + Pf(v+\theta(v)) = \phi_1. \tag{8}$$

Conversely, if (8) holds, then it follows from the definition of $\theta(v)$ that $u = v+\theta(v)$ is a solution of (1). Therefore we may restrict our attention to the two-dimensional problem (8).

The next two lemmas will be very useful in studying the behavior of the mapping $V \to V$ defined by $v \to Av + Pf(v+\theta(v))$.

Lemma 2.2. *There exists a number $m > 0$ such that if $v = s_1\phi_1 + s_2\phi_2$ and*

$|s_2| \leqslant m\,|s_1|$ then $\theta(v)=0$. Moreover, if v satisfies this condition, then $v \geqslant 0$ in Ω if $s_1 \geqslant 0$ and $v \leqslant 0$ in Ω if $s_1 \leqslant 0$.

Proof. Since $\phi_1(x)>0$ for x in Ω and $\partial\phi_1/\partial n<0$ on $\partial\Omega$, there exists a number m such that if $|s| \leqslant m$, then $\phi_1(x)+s\phi_2(x) \geqslant 0$ for all $x \in \Omega$. It follows that if $v=s_1\phi_1+s_2\phi_2$, $|s_2|<m\,|s_1|$, and $s_1 \geqslant 0$, then $v=v^+$ and $v^- \equiv 0$. From (2), we see that $f(v(x))=\beta v^+(x)=\beta v(x)$ on Ω so $(I-P)f \circ v=0$. From this it follows that $w \equiv 0$ is a solution of $Aw+(I-P)f(v+w)=0$, $w \mid \partial\Omega=0$ and, hence, $\theta(v)=0$.

Similarly, if $v=s_1\phi_1+s_2\phi_2$, $|s_2| \leqslant m\,|s_1|$, and $s_1 \leqslant 0$, then $v=v^-$ and $v^+ \equiv 0$ so $(I-P)f \circ v=-(I-P)\,\alpha v=0$. Since $w \equiv 0$ is a solution of $Aw+(I-P)f(v+w)=0$, $w \mid \partial\Omega=0$, $\theta(v)=0$. This proves the lemma.

LEMMA 2.3. *There exists a constant $d>0$ such that if $v=s_1\phi_1+s_2\phi_2$ then $(\phi_1, Av+Pf(v+\theta(v)))_0 \geqslant d\,|s_2|$.*

Proof. First we note that $f(s)-\lambda_1 s=(\beta-\lambda_1)\,s^+ +(\lambda_1-\alpha)\,s^- \geqslant (\lambda_2-\lambda_1)\,s^+ +(\lambda_1-\alpha)\,s^- \geqslant C\,|s|$ where $C=\min(\lambda_2-\lambda_1, \lambda_1-\alpha)>0$. Let $v=s_1\phi_1+s_2\phi_2$. Since $(\phi_1, Av)_0=-s_1\lambda_1=-\lambda_1(\phi_1, v)_0$ we have

$$
\begin{aligned}
(\phi_1, Av+Pf(v+\theta(v)))_0 &= (\phi_1, P(f(v+\theta(v))-\lambda_1 v))_0\\
&= (\phi_1, f(v+\theta(v))-\lambda_1 v)_0\\
&= (\phi_1, f(v+\theta(v))-\lambda_1(v+\theta(v)))_0.
\end{aligned}
$$

Since $\phi_1(x)>0$ on Ω, it follows that $(\phi_1, f(v+\theta(v))-\lambda_1(v+\theta(v)))_0 \geqslant (\phi_1, C\,|v+\theta(v)|)_0=C\int_\Omega \phi_1(x)\,|s_1\phi_1(x)+s_2\phi_2)(x)+\theta(v)(x)|\,dx$.

Referring to Lemma 2.2, we see that there is a constant $m>0$ such that $\phi_1(x) \geqslant m\,|\phi_2(x)|$. Therefore, from the above, we have

$$
\begin{aligned}
&(\phi_1, Av+Pf(v+\theta(v)))_0\\
&\geqslant C\int_\Omega m\,|\phi_2(x)|\,|s_1\phi_1(x)+s_2\phi_2(x)+\theta(v)(x)|\,dx\\
&\geqslant Cm\left|\int_\Omega \phi_2(x)(s_1\phi_1(x)+s_2\phi_2(x)+\theta(v)(x))\,dx\right|=Cm\,|s_2|.
\end{aligned}
$$

Setting $d=Cm$ the assertion of the lemma follows.

We consider the mapping $F: R^2 \to R^2$ defined by $F(s_1, s_2)=(t_1, t_2)$ if $v=s_1\phi_1+s_2\phi_2$ and $Av+Pf(v+\theta(v))=t_1\phi_1+t_2\phi_2$.

PROPOSITION 2.4. *Let $p=(1,0)$. Let r be so large that $1<r$, $1<(\beta-\lambda_1)\,r$, $1<(\lambda_1-\alpha)\,r$, $1<mr$, and $1<mdr$ where m is as in Lemma 2.2 and d is as in Lemma 2.3. Let*

$$D_1 = \{|s_1, s_2)|\ 0 < s_1 < r,\ |s_2| < ms_1\},$$

$$D_2 = \{(s_1, s_2)|\ |s_1| < r,\ m\,|s_1| < s_2 < mr\},$$

$$D_3 = \{(s_1, s_2)|\ -r < s_1 < 0,\ |s_2| < m\,|s_1|\},$$

$$D_4 = \{(s_1, s_2)\ |\ |s_1| < r;\ -mr < s_2 < -m\,|s_1|\}.$$

If $d(F, D_k, p)$ denotes the Brouwer degree of F with respect to D_k and p for $1 \leqslant k \leqslant 4$, then $d(F, D_k, p)$ is defined for $1 \leqslant k \leqslant 4$ and $d(F, D_k, p) = (-1)^{k+1}$.

Remark. If $F(s_1, s_2) = p = (1, 0)$ and $v = s_1\phi_1 + s_2\phi_2$, then $Av + Pf(v + \theta(v)) = \phi_1$ so, according to previous discussion, $u = v + \theta(v)$ is a solution of (1). Since the assertion of the proposition implies the existence of a point $(s_{k1}, s_{k2}) \in D_k$ such that $F(s_{k1}, s_{k2}) = p$ for $k = 1,..., 4$ and since the open sets D_k, $1 \leqslant k \leqslant 4$, are disjoint we infer the existence of four distinct solutions of (1).

Proof of Proposition 2.4. We first consider the region D_1. From Lemma 2.2 we see that if $(s_1, s_2) \in \bar{D}_1$ and $v = s_1\phi_1 + s_2\phi_2$ then $\theta(v) = 0$. Moreover, referring to the proof of Lemma 2.2, we see that if $(s_1, s_2) \in D_1$ then $s_1\phi_1 + s_2\phi_2 \geqslant 0$ on Ω. Therefore, if $(s_1, s_2) \in \bar{D}_1$, then

$$\begin{aligned} A(s_1\phi_1 + s_1\phi_2) &+ Pf(s_1\phi_1 + s_2\phi_2 + \theta(s_1\phi_1 + s_2\phi_2)) \\ &= -(\lambda_1 s_1\phi_1 + \lambda_2 s_2\phi_2) + P(\beta(s_1\phi_1 + s_2\phi_2)) \\ &= (\beta - \lambda_1)\, s_1\phi_1 + (\beta - \lambda_2)\, s_2\, \phi_2 \end{aligned}$$

so $F(s_1, s_2) = ((\beta - \lambda_1)\, s_1,\ (\beta - \lambda_2)\, s_2)$. Since $1 < r(\beta - \lambda_1)$ the equation $F(s_1, s_2) = (1, 0)$ has the unique solution $(s_1, s_2) = ((\beta - \lambda_1)^{-1}, 0)$ for $(s_1, s_2) \in D_1$. Since the determinant of the linear map $(s_1, s_2) \to ((\beta - \lambda_1)\, s_1,$ $(\beta - \lambda_2)\, s_2)$ is $(\beta - \lambda_1)(\beta - \lambda_2) > 0$, it follows that $d(F, D_1, p) = 1$.

Similarly, if $(s_1, s_2) \in \bar{D}_3$ then, from the proof of Lemma 2.2, we see that $s_1\phi_1 + s_2\phi_2 \leqslant 0$ in Ω and $\theta(s_1\phi_1 + s_2\phi_2) = 0$. Therefore, $A(s_1\phi_1 + s_2\phi_2) + Pf(s_1\phi_1 + s_2\phi_2 + \theta(s_1\phi_1 + s_2\phi_2)) = -(\lambda_1 s_1\phi_1 + \lambda_2 s_2\phi_2) + \alpha(s_1\phi_1 + s_2\phi_2) = (\alpha - \lambda_1)\, s_1\phi_1 + (\alpha - \lambda_2)\, s_2\phi_2$, so $F(s_1, s_2) = ((\alpha - \lambda_1)\, s_1,\ (\alpha - \lambda_2)\, s_2)$.

It follows that for $(s_1, s_2) \in \bar{D}_3$ the equation $F(s_1, s_2) = (1, 0)$ has the unique solution $(s_1, s_2) = ((\alpha - \lambda_1)^{-1}, 0)$. Since the determinant of the linear map $(s_1, s_2) \to ((\alpha - \lambda_1)\, s_1,\ (\alpha - \lambda_2)\, s_2)$ is $(\alpha - \lambda_1)(\alpha - \lambda_2) > 0$, we see that $d(F, D_3, p) = 1$.

To calculate $d(F, D_2, p)$ we consider the mapping $H: \bar{D}_2 \times [0, 1] \to R^2$ defined by $H(s_1, s_2, \tau) = (1 - \tau)\, F(s_1, s_2) + \tau(s_2, s_1)$. We shall show that $H(s_1, s_2, \tau) \neq (1, 0)$ for all $(s_1, s_2) \in \partial D_2$ and $\tau \in [0, 1]$ and use the homotopy invariance theorem to calculate $d(F, D_2, p)$. First, if $0 \leqslant s_1 \leqslant r$ and $s_2 = ms_1$ then $(s_1, s_2) \in \partial D_1$ so, from what has been shown,

$$F(s_1, s_2) = ((\beta - \lambda_1) s_1, (\beta - \lambda_2) s_2)$$

and

$$H(s_1, s_2, \tau) = ((1-\tau)(\beta - \lambda_1) s_1 + \tau m s_1, (1-\tau)(\beta - \lambda_2) m s_1 + \tau s_1).$$

If $0 < s_1 \leqslant r$ and $0 \leqslant \tau \leqslant 1$, then $(1-\tau)(\beta - \lambda_2) m s_1 + \tau s_1 > 0$. Therefore, from the above, we see that $H(s_1, s_2, \tau) \neq (1, 0) = p$ for $0 \leqslant s_1 \leqslant r$, $s_2 = ms_1$, and $0 \leqslant \tau \leqslant 1$.

Second, if $-r \leqslant s_1 \leqslant 0$ and $s_2 = -ms_1$, then $(s_1, s_2) \in \partial D_3$ so, according to what has already been shown, $F(s_1, s_2) = ((\alpha - \lambda_1) s_1, (\alpha - \lambda_2) s_2)$. Consequently, for $0 \leqslant \tau \leqslant 1$, $H(s_1, s_2, \tau) = ((1-\tau)(\alpha - \lambda_1) s_1 - \tau m s_1, -(1-\tau)(\alpha - \lambda_2) m s_1 + \tau s_1)$. Since $-(1-\tau)(\alpha - \lambda_2) m s_1 + \tau s_1 < 0$ for $-r \leqslant s_1 < 0$ and $0 \leqslant \tau \leqslant 1$, we see that $H(s_1, s_2, \tau) \neq (1, 0)$ for $-r \leqslant s_1 \leqslant 0$, $s_2 = -ms_1$, and $0 \leqslant \tau \leqslant 1$.

Finally, we consider the portion of ∂D_2 consisting of the segment $-r \leqslant s_1 \leqslant r$, $s_2 = mr$. Let $F(s_1, s_2) = (F_1(s_1, s_2), F_2(s_1, s_2))$ and $H(s_1, s_2, \tau) = (H_1(s_1, s_2, \tau), H_2(s_1, s_2, \tau))$. If $s_2 = mr$, then, according to Lemma 2.3, $F_1(s_1, s_2) \geqslant dmr$. Hence, for $0 \leqslant \tau \leqslant 1$, $H_1(s_1, s_2, \tau) = (1-\tau) F_1(s_1, s_2) + \tau s_2 \geqslant (1-\tau) dmr + \tau mr$. Since $1 < dmr$ and $1 < mr$, it follows that $H(s_1, s_2, \tau) \neq (1, 0)$ for $-r \leqslant s_1 \leqslant r$, $s_2 = mr$, and $0 \leqslant \tau \leqslant 1$.

We have shown that $H(s_1, s_2, \tau) \neq (1, 0)$ for $(s_1, s_2) \in \partial D_2$ and $0 \leqslant \tau \leqslant 1$. By homotopy invariance of degree, if $G(s_1, s_2) = H(s_1, s_2, 1) = (s_2, s_1)$, then $d(F, D_2, p) = d(G, D_2, p)$. Since $1 < mr$, the equation $G(s_1, s_2) = (1, 0) = p$ has a unique solution in D_2, namely $(s_1, s_2) = (0, 1)$. As the determinant of the linear map $(s_1, s_2) \to (s_2, s_1)$ is equal to -1, we see that $d(F, D_2, p) = d(G, D_2, p) = -1$.

To calculate $d(F, D_4, p)$ we consider the homotopy $\hat{H}: \bar{D}_4 \times [0, 1] \to R^2$ defined by $\hat{H}(s_1, s_2, \tau) = (1-\tau) F(s_1, s_2) + \tau(-s_2, -s_1)$. We consider a point $(s_1, s_2) \in \partial D_4$. If (s_1, s_2) is on the segment $0 \leqslant s_1 \leqslant r$, $s_2 = -ms_1$, then $(s_1, s_2) \in \partial D_1$ so, according to what has been shown, $\hat{H}(s_1, s_2, \tau) = ((1-\tau)(\beta - \lambda_1) s_1 + \tau m s_1, (1-\tau)(\beta - \lambda_2)(-ms_1) - \tau s_1)$. Since $-(1-\tau)(\beta - \lambda_2) m s_1 - \tau s_1 < 0$ for $0 \leqslant \tau \leqslant 1$ and $0 < s_1 \leqslant r$, it follows that $\hat{H}(s_1, s_2, \tau) \neq (1, 0)$ for $0 \leqslant \tau \leqslant 1$, $0 \leqslant s_1 \leqslant r$, and $s_2 = -ms_1$. If (s_1, s_2) is on the segment $-r \leqslant s_1 \leqslant 0$, $s_2 = ms_1$, then $(s_1, s_2) \in \partial D_3$. Therefore, $\hat{H}(s_1, s_2, \tau) = ((1-\tau)(\alpha - \lambda_1) s_1 - \tau m s_1, (1-\tau)(\alpha - \lambda_2) m s_1 - \tau s_1)$. Since $(1-\tau)(\alpha - \lambda_2) m s_1 - \tau s_1 > 0$ for $0 \leqslant \tau \leqslant 1$ and $-r \leqslant s_1 < 0$, it follows that $\hat{H}(s_1, s_2, \tau) \neq (1, 0)$ for $0 \leqslant \tau \leqslant 1$, $-r \leqslant s_1 \leqslant 0$, and $s_2 = ms_1$. Setting $\hat{H}(s_1, s_2, \tau) = (\hat{H}_1(s_1, s_2, \tau), \hat{H}_2(s_1, s_2, \tau))$, we see from Lemma 2.3 that if (s_1, s_2) is on the segment where $|s_1| \leqslant r$ and $s_2 = -mr$, then for $0 \leqslant \tau \leqslant 1$,

$\hat{H}_1(s_1, s_2, \tau) \geqslant (1-\tau)\,dmr + \tau mr > 1$. Hence, $\hat{H}(s_1, s_2, \tau) \neq (1, 0)$ for all $(s_1, s_2, \tau) \in \partial D_4 \times [0, 1]$. Consequently, if $G(s_1, s_2) = \hat{H}(s_1, s_2, 1) = (-s_2, -s_1)$, then $d(F, D_4, p) = d(G, D_4, p)$. Since $-1 > -mr$ the equation $G(s_1, s_2) = (1, 0) = p$ has the unique solution $(s_1, s_2) = (0, -1)$ in D_4. As the determinant of the linear map $(s_1, s_2) \to (-s_2, -s_1)$ is -1, we see that $d(G, D_4, p) = -1$ and the proposition is proved.

Let D_k, $k = 1, \ldots, 4$, and F be as above. Let M be the linear map defined by $M(s_1, s_2) = (-s_1/\lambda_1, -s_2/\lambda_2)$ since the determinant of M is $1/\lambda_1\lambda_2 > 0$, it follows that if $\hat{p} = Mp = (-1/\lambda_1, 0)$, then $d(M \circ F, D_k, \hat{p}) = (-1)^{k+1}$ for $1 \leqslant k \leqslant 4$. If $v = s_1\phi_1 + s_2\phi_2 \in V$, then $A^{-1}(v) = (-s_1/\lambda_1)\,\phi_1 + (-s_2/\lambda_2)\,\phi_2$ and from this we see that $M \circ F(s_1, s_2) = (t_1, t_2)$ if and only if $v + A^{-1}Pf(v + \theta(v)) = t_1\phi_1 + t_2\phi_2$. Using the definition of degree of a mapping on an arbitrary finite dimensional vector space (see, for example, [14, p. 105]), we obtain

PROPOSITION 2.5. *If for* $1 \leqslant k \leqslant 4$, $U_k = \{v \in V \mid v = s_1\phi_1 + s_2\phi_2, (s_1, s_2) \in D_k\}$ *and* $T\colon v \to v$ *is defined by* $Tv = PA^{-1}f(v + \theta(v))$ *then* $d(I + T, U_k, -\phi_1/\lambda_1) = (-1)^{k+1}$.

(Here I is the identity on V and we have used the fact that $PA^{-1} = A^{-1}P$.)

To prove the main result we make use of a lemma very similar to Lemma 5 of [8]. First we recall that A^{-1} extends as a map from $C^\gamma(\bar{\Omega})$ onto the space of functions in $C^{2+\gamma}(\bar{\Omega})$ which vanish on $\partial\Omega$ to a continuous, one-to-one, linear mapping from $L^2(\Omega)$ onto $H_2(\Omega) \cap H_1^0(\Omega)$. Since the Nemytskii map $u \to f \circ u$ maps $L^2(\Omega)$ continuously into $L^2(\Omega)$ and the embedding of $H_2(\Omega)$ into $L^2(\Omega)$ is compact, the mapping $N\colon L^2(\Omega) \to L^2(\Omega)$ defined by $N(u) = A^{-1}(f \circ u)$ is compact and continuous.

LEMMA 2.6. *Let* U_k, $1 \leqslant k \leqslant 4$, *and T be as in Lemma* 2.5. *If* $r_2 > 0$ *is sufficiently large and for* $1 \leqslant k \leqslant 4$

$$Y_k = \{u \in L^2(\Omega) \mid Pu \in U_k, \|(I-P)\,u\| < r_2\} \tag{9}$$

then the Leray–Schauder degree $d(I + N, Y_k, -\phi_1/\lambda_1)$ *is defined and* $d(I + N, Y_k, -\phi_1/\lambda_1) = d(I + T, U_k, -\phi_1/\lambda_1) = (-1)^{k+1}$.

Proof. We first show that there exists a number r_1 such that if $v \in \bar{U}_k$, $1 \leqslant k \leqslant 4$, $0 \leqslant s \leqslant 1$, and $w = -(1-s)(I-P)\,N(v+w)$, then $\|w\| \leqslant r_1$. In fact, if $Aw + (1-s)(I-P)f(v+w) = 0$, $v \in V$, and $w \in W$, then $(A-b)(w) + (I-P)[(1-s)f(v+w) + b(v+w)] = 0$ where b is as in the proof of Lemma 2.1. If $g(t) = (1-s)f(t) + bt$, then g satisfies a Lipshitz condition with Lipshitz constant $(1-s)\,\beta + b \leqslant \beta + b$. Since $\|(A-b)^{-1}(I-P)\| = (\lambda_3 + b)^{-1}$, we have $\|w\| \leqslant \|(A-b)^{-1}(I-P)$

$(g(v+w)-(g(v))\| + \|(A-b)^{-1}(I-P)\ g(v)\| \leqslant \delta\|w\| + (\lambda_3+b)^{-1}\ \|g(v)\|$, where $\delta=(\beta+b)/\ (\lambda_3+b)<1$. Therefore, if r_1 is chosen so that $(\lambda_3+b)^{-1}$ $(1-\delta)^{-1}(\|f(v)\|+b\|v\|)\leqslant r_1$ for all $v\in \bar{U}_k$, $1\leqslant k\leqslant 4$, then $\|w\|\leqslant r_1$ and the claim is established.

We shall show that if $r_2>r_1$ and Y_k $(1\leqslant k\leqslant 4)$ is defined by (9), then the assertion of the lemma follows. To this end, let k be fixed with $1\leqslant k\leqslant 4$ and let r_2 satisfy this inequality. We define h_1: $Y_k\times[0,1]\to L^2(\Omega)$ by $h_1(u,s)=(I-P)\,N(v+w)-PN(v+w+s(\theta(v)-w))$, where $v=Pu$, $w=(I-P)\,u$. We claim that $u+h_1(u,s)\neq -\phi_1/\lambda_1$ for $(u,s)\in\partial Y_k\times[0,1]$. There are two possibilities to consider. First, suppose that $u=v+w$ where $v\in\partial U_k$, $w\in W$ with $\|w\|<r_2$, $s\in[0,1]$, and $u+h_1(u,s)=-\phi_1/\lambda_1$. It follows that $w+(I-P)\,N(v+w)=0$ and $v+PN(v+w+s(\theta(v)-w))=-\phi_1/\lambda_1$. The first of these equations implies that $w=\theta(v)$ and hence, from the second, we have $v+PN(v+\theta(v))=v+T(v)=-\phi_1/\lambda_1$, which is a contradiction since $v\in\partial U_k$. This shows that $u+h_1(u,s)\neq-\phi_1/\lambda_1$ if $Pu\in\partial U_k$ and $0\leqslant s\leqslant 1$. Second, suppose that $u=v+w$ where $v\in U_k$, $w\in w$ and $\|w\|=r_2$. If $0\leqslant s\leqslant 1$ and $u+h_1(u,s)=-\phi_1/\lambda_1$, then $w+(I-P)$ $N(v+w)=0$, so, according to what has been shown, $w=\theta(v)$ and $\|w\|\leqslant$ $r_1<r_2$, which is a contradiction. This shows that $u+h_1(u,s)\neq-\phi_1/\lambda_1$ for all $(u,s)\in\partial Y_k\times[0,1]$. Since $h_1(u,0)=N(u)$, it follows by homotopy invariance of degree that

$$d(I+N,\ Y_k,\ -\phi_1/\lambda_1)=d(I+h_1(\cdot,1),\ Y_k,\ -\phi_1/\lambda_1). \tag{10a}$$

Let h_2: $Y_k\times[0,1]\to L^2(\Omega)$ be defined by $h_2(u,s)=(1-s)$ $(I-P)\,N(u)+PN(v+\theta(v))$ where $v=Pu$. If $v\in\partial U_k$, $w\in W$, $0\leqslant s\leqslant 1$, $u=v+w$, and $u+h_2(u,s)=-\phi_1/\lambda_1$, then $v+T(v)=v+PN(v+\theta(v))=P(u+h_2(u,s))=-\phi_1/\lambda_1$ which is a contradiction. Therefore, $u+h_2(u,s)\neq-\phi_1/\lambda_1$ if $Pu\in\partial U_k$ and $0\leqslant s\leqslant 1$. If $u=v+w$ with $v\in U_k$, $w\in w$, $\|w\|=r_2$, and $u+h_2(u,s)=-\phi_1/\lambda_1$, then $0=(I-P)(u+h_2(u,s))=w+(1-s)(I-P)\ N(v+w)$. From an estimate established above, it follows that $\|w\|\leqslant r_1<r_2$, which is a contradiction. This shows that $u+h_2(u,s)\neq$ $-\phi_1/\lambda_1$ for $(u,s)\in\partial Y_k\times[0,1]$. Since $h_1(u,1)=h_2(u,0)$, we infer by homotopy invariance and (10a) that $d(I+N,\ Y_k,\ -\phi_1/\lambda_1)=$ $d(I+h_2(\cdot,1),\ Y_k,\ -\phi_1/\lambda_1)$. Let B be the open ball of radius r_2 in W. If $u\in\bar{Y}_k$, $v=Pu$, and $w=(I-P)\,u$, then $u+h_2(u,1)=v+PN(v+\theta(v))+w$. Hence, by the product property of degree (see [13, p. 11]) $d(I+N,\ Y_k,$ $-\phi_1/\lambda_1)\ =\ d(I,\ B,\ 0)d(I+T,\ U_k,\ -\phi_1/\lambda_1)\ =\ d(I+T,\ U_k,\ -\phi_1/\lambda_1)\ =$ $(-1)^{k+1}$. This proves the lemma.

3. The Full Nonlinear Elliptic Problem

Let $g: R \to R$ be a C^1 function which satisfies the asymptotic conditions $\lim_{\xi \to -\infty}(g(\xi)/\xi) = \alpha$, $\lim_{\xi \to \infty}(g(\xi)/\xi) = \beta$ where α and β are as in the previous section. If, as before, we set $f(\xi) = \beta\xi^+ - \alpha\xi^-$ we can write

$$g(\xi) = f(\xi) + f_0(\xi)$$

where

$$\lim_{|\xi| \to \infty} f_0(\xi)/\xi = 0. \tag{10b}$$

We consider the boundary value problem

$$Au + g(u) = s\phi_1(x) + h(x) \text{ in } \Omega, \qquad u \mid \partial\Omega = 0 \tag{11}$$

where $s > 0$ and $h \in C^\gamma(\bar{\Omega})$ satisfies $(h, \phi_1)_0 = 0$. Although the function f is nonlinear, it satisfies $f(\xi)/s = f(\xi/s)$ for $s > 0$. Therefore, if we set $u = sz$ in (11), we arrive at the problem

$$Az + f(z) + f_0(sz)/s = \phi_1(x) + \frac{h(x)}{s} \text{ in } \Omega, \qquad z \mid \partial\Omega = 0. \tag{12}$$

In view of the condition (10b) we consider (12) as a perturbation of the piecewise linear problem (1). If, as before, we consider $A^{+1}: L^2(\Omega) \to H_1^0 \subset L^2(\Omega)$ as a compact mapping from $L^2(\Omega)$ into itself, then (12) is equivalent to the operator equation $z + N_s(z) = -\phi_1/\lambda_1$ where $N_s: L^2(\Omega) \to L^2(\Omega)$ is the compact and continuous mapping defined by $N_s(z) = A^{-1}[f(z) + f_0(sz)/s - h/s]$. Let N have the same meaning as in Lemma 2.6.

Lemma 3.1.

$$\lim_{s \to \infty} \|N(z) - N_s(z)\| = 0 \tag{13}$$

uniformly for z in bounded subsets of $L^2(\Omega)$.

Proof. From (10b) we see that for any $\varepsilon > 0$ there exists a number $\xi_0 = \xi_0(\varepsilon) > 0$ such that $|f_0(\xi)| \leqslant \varepsilon |\xi|$ if $|\xi| \geqslant \xi_0(\varepsilon)$. Setting $C(\varepsilon) = \max_{|\xi| \leqslant \xi_0} |f_0(\xi) - \varepsilon |\xi| \,|$, it follows that $|f_0(\xi)| \leqslant \varepsilon |\xi| + C(\varepsilon)$ for all ξ. Therefore, if $z \in L^2(\Omega)$, $\|N_s(z) - N(z)\| = \|A^{-1}(f_0(sz)/s - h/s)\| \leqslant (1/s)\, \|A^{-1}\| (\varepsilon \|sz\| + C(\varepsilon) + \|h\|)$, so $\|Ns(z) - N(z)\| \leqslant \varepsilon \|A^{-1}\| (\|z\| + 1)$ if $s \geqslant (C(\varepsilon) + \|h\|)/\varepsilon$. This proves the result.

We now prove the main result of this section.

THEOREM 3.1. *Let f be as in* (2). *If* (3) *and* (10b) *hold, there exists s_0 such that for $s > s_0$ the boundary value problem* (11) *has at least four solutions. More generally, s_0 can be chosen so that there exists a number $\sigma > 0$ such that if $b_k, k = 1,..., n$, are smooth functions defined on Ω with $|b_k|_\infty \leqslant \sigma$ for $k = 1,..., n$ and A_1 is the nonselfadjoint operator defined by*

$$A_1 u = Au + \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k} \tag{14}$$

then for $s > s_0$ the boundary value problem

$$A_1 u + g(u) = s\phi_1(x) + h(x), \qquad u \mid \partial\Omega = 0 \tag{15}$$

has at least four solutions.

Proof. Referring to the proof of Lemma 2.6, we see that $z + N(z) \neq -\phi_1/\lambda_1$ for all $z \in \partial Y_k$, $1 \leqslant k \leqslant 4$ (this is also implicit in the fact that $d(I + N, Y_k, -\phi_1/\lambda_1)$ is defined). Since ∂Y_k is closed and bounded for $1 \leqslant k \leqslant 4$ and N is continuous and compact there exists a number $\eta > 0$ such that $\|z + N(z) + \phi_1/\lambda_1\| \geqslant \eta$ if $z \in \partial Y_k$, $1 \leqslant k \leqslant 4$. According to Lemma 3.4, there exists $s_0 > 0$ such that $\|N_s(z) - N(z)\| < \eta/2$ for $z \in \partial Y_k$, $1 \leqslant k \leqslant 4$. It follows that

$$\|z + N_s(z) + \phi_1/\lambda_1\| \geqslant \eta/2 \tag{15}$$

for $s > s_0$ and $z \in \partial Y_k$, $1 \leqslant k \leqslant 4$. Thus, by the Poincare–Bohl theorem of degree theory, if $s > s_0$, then

$$d(I + N_s, Y_k, -\phi_1/\lambda_1) = d(I + N, Y_k, -\phi_1/\lambda_1) = (-1)^{k+1}, \qquad 1 \leqslant k \leqslant 4. \tag{16}$$

If b_k, $1 \leqslant k \leqslant n$, are smooth functions and A_1 is defined as in the statement of the theorem, then, by standard elliptic theory, A_1 is a one-to-one continuous mapping from the subspace of functions in $C^{2+\gamma}(\bar{\Omega})$ which vanish on $\partial\Omega$ onto $C^\gamma(\bar{\Omega})$ and A_1^{-1} extends from $L^2(\Omega)$ onto $H_1^0(\Omega)$. We estimate $\|A^{-1} - A_1^{-1}\|$ in terms of $\max_{1 \leqslant k \leqslant n} |b_k|_\infty$ where both A^{-1} and A_1^{-1} are considered as compact mappings from $L^2(\Omega)$ into itself.

Let z be smooth and suppose that $A_1 u = z$ and $u \mid \partial\Omega = 0$. We have

$$u = A^{-1}\left(z - \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k}\right). \tag{17}$$

For each $k = 1,..., n$ the $L^2(\Omega)$-norm of $\partial u/\partial x_k$ is less or equal to the $H_1^0(\Omega)$-norm of u which in turn is less or equal to $\|A_1^{-1} z\|$. Therefore, if we set $\omega = \max_{1 \leqslant k \leqslant n} |b_k|_\infty$, then $\|A_1^{-1} z\| = \|u\| \leqslant \|A^{-1}\|(\|z\| + n\omega \|A_1^{-1} z\|)$. Thus assuming $n\omega \|A^{-1}\| < 1$, we have $\|A_1^{-1}\| \leqslant (1 - n\omega \|A^{-1}\|)^{-1} \|A^{-1}\|$. Returning to (17), we have $\|A_1^{-1}z - A^{-1}z\| \leqslant \|A^{-1}\| \|A_1^{-1}\| n\omega \|z\|$. Hence,

$$\|A_1^{-1} - A^{-1}\| \leqslant \frac{\|A^{-1}\|^2 n\omega}{1 - n\omega \|A^{-1}\|} \tag{18}$$

Setting $u = sz$ in (15), where $s > 0$, we obtain

$$A_1 z + f(z) + f_0 \frac{(sz)}{s} = \phi_1 + \frac{h(x)}{s}, \qquad z \mid \partial\Omega = 0. \tag{19}$$

This is equivalent to the operator equation

$$z + \hat{N}_s(z) = A_1^{-1} \phi_1 \tag{20}$$

where $\hat{N}_s(z) = A_1^{-1}[f_z + f_0(sz)/s - h/s]$. By (18), we infer the existence of a number $\sigma > 0$ such that if $\max_{1 \leqslant k \leqslant n} |b_k|_\infty \leqslant \sigma$, then

$$\begin{aligned} &\|(N_s(z) + \phi_1/\lambda_1) - (\hat{N}_s(z) - A_1^{-1} \phi_1\| \\ &\leqslant \|A^{-1} - A_1^{-1}\| \, \|f(z) - f_0(sz)/s - h/s\| < \eta/2 \end{aligned} \tag{21}$$

for $s > s_0$ and $z \in \partial Y_k$, $1 \leqslant k \leqslant 4$, where η is as in (15). By the Poincare–Bohl theorem, it follows from (15) and (21) that $d(I + \hat{N}_s, Y_k, A_1^{-1} \phi_1) = d(I + \hat{N}_s - A_1^{-1} \phi_1, Y_k, 0) = d(I + N_s + \phi_1/\lambda_1, Y_k, 0) = d(I + N_s, Y_k, -\phi_1/\lambda_1) = (-1)^{k+1}$ for $k = 1, \ldots, 4$. Therefore for each $k = 1, \ldots, 4$ there exists a solution z_k of (20) with $z_k \in Y_k$. By a bootstrap argument, these solutions are smooth and satisfy (19). Setting $u_k = sz_k$ for $s > s_0$ and $1 \leqslant k \leqslant 4$, we obtain four distinct solutions of (15) and the theorem is proved.

4. Parabolic Equations

In this section we consider distributional solutions of the periodic boundary value problem

$$u_{xx} - u_t + g(u) = s\phi_1(x) + h(t, x), \qquad 0 < x < \pi,\ -\infty < t < \infty \tag{22}$$

$$u(t, 0) = u(t, \pi) = 0, \qquad -\infty < t < \infty \tag{23}$$

$$u(t + T, x) \equiv u(t, x), \qquad T > 0,\ -\infty < t < \infty, 0 \leqslant x \leqslant \pi. \tag{24}$$

Here $g(\xi) = f(\xi) + f_0(\xi)$, where f has the form (2) with α and β to be specified below, f_0 is a continuous function satisfying the asymptotic condition (10), $\phi_1(x) = \sqrt{2/\pi T} \sin x$, and h is continuous on $R \times [0, \pi]$ and satisfies $h(t+T, x) \equiv h(t, x)$ and $\int_0^T (\int_0^\pi h(t, x) \sin x \, dx) \, dt = 0$. The number T is required to satisfy

$$0 < T < 2\pi/\sqrt{15} \tag{25}$$

and the numbers α and β are required to satisfy

$$0 \leqslant \alpha < 1, \qquad 4 < \beta < \min(9, \sqrt{1 + 4\pi^2/T^2}). \tag{26}$$

We use the definition of distributional solution given in Chapter 45 of [5]. To state this definition we need some preliminary concepts. Let $Q = R \times (0, \pi)$. Let $H_T^0(Q)$ denote the set of all real measurable functions $u(t, x)$ defined on Q such that $u(t+T, x) = u(t, x)$ almost everywhere on Q and u is square integrable on $(0, T) \times (0, \pi)$. If for u and $v \in H_T^0(Q)$ we set $\langle u, v \rangle_0 = \int_0^T (\int_0^\pi u(x, t) \, v(x, t) \, dx) \, dt$, then $H_T^0(Q)$ is a real Hilbert space with inner product $\langle \, , \, \rangle_0$.

Let $H_T^1(Q)$ and $H_T^{1,2}(Q)$ denote the completion of the real inner product spaces consisting of functions which are infinitely differentiable on $\bar{Q}$ and T-periodic in t with inner products

$$\langle u, v \rangle_1 \equiv \langle u, v \rangle_0 + \langle u_x, v_x \rangle_0 + \langle u_t, v_t \rangle_0 \tag{27}$$

and

$$\langle u, v \rangle_{1,2} \equiv \langle u, v \rangle_1 + \langle u_{xx}, v_{xx} \rangle_0 \tag{28}$$

respectively. It is clear that a member u of $H_T^1(Q)(H_T^{2,1}(Q))$ has distributional derivatives $u_t, u_x \in H_T^0(Q)$ $(u_t, u_x, u_{xx} \in H_T^0(Q))$ and these distributional derivatives can be obtained as limits in $L^2((0, T) \times (0, \pi))$ of the corresponding derivatives of a sequence of smooth functions which tend to u in $H_T^1(Q)(H_T^{1,2}(Q))$. Moreover, if derivatives are interpreted as distributional derivatives, then the inner products in $H_T^1(Q)$ and $H_T^{1,2}(Q)$ are given by (27) and (28), respectively.

It is known (see [12]) that if $C_T(\bar{Q})$ denotes the set of functions which are continuous on Q and T-periodic in t, then $H_T^{1,2}(Q) \subset C_T(\bar{Q})$, and the embedding of $H_T^{1,2}(Q)$ in $C_T(\bar{Q})$ is continuous. Also, the embeddings of $H_T^{1,2}(Q)$ and $H_T^1(Q)$ in $H_T^0(Q)$ are compact.

In the following we denote by $\mathring{H}_T^1(Q)$ the closure in $H_T^1(Q)$ of functions $u(t, x)$ which are infinitely differentiable on Q, T-periodic in t, and satisfy the boundary conditions $u(t, 0) = u(t, \pi) = 0$, $t \in R$.

By a weak solution of (22)–(24) we mean a function $u \in H_T^{1,2}(Q) \cap \mathring{H}_T^1(Q)$ which satisfies (22) almost everywhere. (This is the definition given in [5, Chap. 45].) We show that the techniques of the previous two sections can be used to prove the following.

THEOREM 4.1. *If* (25) *and* (26) *hold then there exists a number* s_0 *such that if* $s > s_0$ *then the problem* (22)–(24) *has at least four solutions.*

The proof of this theorem is based on several lemmas. Since the proofs of these lemmas are completely analogous to the proofs of corresponding lemmas in the previous sections we only give the first one completely.

Let $Lu = u_{xx} - u_t$ for $u \in H_T^{1,2}(Q)$. Given $h \in H_T^0(Q)$ there exists a unique $u \in H_T^{1,2}(Q) \cap H_T^1(Q)$ such that $Lu = h$. This can be proved by noting that if h has a representation

$$\begin{gathered} h(t, x) = \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} C_{k,m}(\sin kx)\, e^{2\pi imt/T}, \\ C_{k,m} = C_{k,-m}, \qquad k \geqslant 1, \quad -\infty < m < \infty \end{gathered} \tag{29}$$

where the series converges to h in $L^2((C, C+T) \times (0, \pi))$ (C arbitrary), then $Lu = h$ has a unique solution $u \in H_T^{1,2}(Q) \cap \mathring{H}_T^1(Q)$ with the representation

$$u(t, x) = \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{C_{k,m}(\sin kx)\, e^{2\pi imt/T}}{-(k^2 + 2\pi im/T)}. \tag{30}$$

Since the mapping $L^{-1}\colon H_T^0(Q) \to H_T^{1,2}(Q)$ is continuous and the embedding from $H_T^{1,2}(Q)$ into $H_T^0(Q)$ is compact, we may regard L^{-1} as a compact linear mapping of $H_T^0(Q)$ into itself.

Let $\phi_1(x)$ be as above and let $\phi_2(x) = \sqrt{2/\pi T}\sin 2x$. We denote by V the subspace of $H_T^0(Q)$ spanned by ϕ_1 and ϕ_2 and by W the orthogonal complement of V in $H_T^0(Q)$. We denote the orthogonal projection of $H_T^0(Q)$ onto V by P.

LEMMA 4.2. *Given* $v \in V$ *there exists a unique* $\theta(v) \in W\ \mathring{H}_T^1(Q) \cap H_T^{1,2}(Q)$ *such that* θ *is continuous and*

$$L\theta(v) + (I-P) f(v + \theta(v)) = 0. \tag{31}$$

Proof. If h has the representation (29) and T denotes the set of pairs of integers (k, m) such that $k \geqslant 1$, $-\infty < m < \infty$, and $(k, m) \neq (1, 0)$ or $(2, 0)$, then in $H_T(Q)$, $(I-P)\, h(t, x) = \sum_{(k,m)} \sum_{\varepsilon J} C_{k,m}(\sin kx)\, e^{2\pi imt/T}$. If $\| \ \|$ denotes the norm of $H_T(Q)$ defined by $\langle\ ,\ \rangle_0$ then $\|h\|^2 = (T\pi/2) \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} |C_{k,m}|^2$. According to (30),

$$L^{-1}(I-P)\, h(x, t) = \sum_{(k,m)} \sum_{\varepsilon J} \frac{C_{k,m}(\sin kx)\, e^{2\pi imt/T}}{-(k^2 + 2\pi im/T)}$$

so

$$\|L^{-1}(I-P)h\|^2 = \frac{\pi T}{2} \sum_{(k,m)} \sum_{\epsilon J} \frac{|C_{k,m}|^2}{k^4 + 4\pi^2 m^2/T^2}.$$

Therefore $\|L^{-1}(I-P)h\| \leqslant (1/\min(9, \sqrt{1+4\pi^2/T^2}))\,\|h\|$. Since $\alpha \geqslant 0$ and $f(s) = \beta s^+ - \alpha s^-$, $|f(s_1) - f(s_2)| \leqslant \beta\,|s_1 - s_2|$ for arbitrary $s_1, s_2 \in R$. Hence, if $v \in V$ and w_1, $w_2 \in W$, then $\|L^{-1}(I-P)(f(v+w_1) - f(v+w_2))\| \leqslant \delta\,\|w_1 - w_2\|$ where $\delta = \beta/\min(9, \sqrt{1+4\pi^2/T^2}) < 1$. Since $L^{-1}(I-P) = (I-P)L^{-1}$, for fixed $v \in V$, the mapping $w \to -L^{-1}(I-P)f(v+w)$ takes W into W. By the contraction mapping principle, there exists a unique $\theta(v) \in W$ such that $\theta(v) = -L^{-1}(I-P)f(v+\theta(v))$, or equivalently, $\theta(v)$ satisfies (31). The continuity of θ follows as in the proof of Lemma 2.1. This proves the result.

We consider the reduced problem

$$Lu + f(u) = \phi_1 \qquad u \in H_T^{1,2}(Q) \cap H_T^1(Q). \tag{32}$$

The same reasoning that was used in the second section shows that u is a solution of (32) if and only if $u = v + w$ with $v \in v$, $w = \theta(v)$, and

$$Lv + Pf(v + \theta(v)) = 0. \tag{33}$$

Since $\phi_1(x) > 0$ on $(0, \pi)$, $\phi_2'(0) > 0$ and $\phi_2'(\pi) < 0$ and ϕ_1 and ϕ_2 both vanish at 0 and π, there exists a number $m > 0$ such that if $|s| \leqslant m$ then $\phi_1(x) + s\phi_2(x) \geqslant 0$ for all $x \in [0, \pi]$.

The proof of Theorem 4.1 follows from this observation, Lemma 4.2, and the exact same reasoning used for the elliptic case.

5. Existence of a Fifth Solution in the Elliptic Case

Let A be as in Section 2, let f have the form (2), and let $g = f + f_0$ where f_0 satisfies (10b). In this section α will denote a fixed number with $\alpha < \lambda_1$ and we will again assume λ_2 to be simple. We will show that there at least four solutions of (11) when $\beta = \lambda_3$ and s is sufficiently large. It will then be shown that if λ_3 has odd multiplicity, there exists $\varepsilon > 0$ such that $\lambda_3 < \beta < \lambda_3 + \varepsilon$ implies the existence of at least five solutions for s sufficiently large.

Throughout this section ϕ_1, ϕ_2, V, W, and P will have the same meanings as in Sections 2 and 3. We will make use of the results of these sections and some additional lemmas.

Lemma 5.1. *If $\lambda_1 < \beta' < \beta''$, then there exists r_3 such that if $\beta' \leqslant \beta \leqslant \beta''$*

and $Au+\beta u^{+}-\alpha u^{-}=\phi_1$ *in* Ω, $u \mid \partial\Omega=0$, *then* $\|u\|<r_3$. (As before, $\| \ \|$ is the $L^2(\Omega)$ norm.)

Proof. Assuming the contrary, there exists a sequence $\{\beta_k\}_1^\infty$ and a corresponding sequence of smooth functions $\{u_k\}_1^\infty$ such that $Au_k+\beta_k u_k^{+}-\alpha u_k^{-}=\phi_1$ in Ω, $u_k \mid \partial\Omega=0$, $\beta'\leqslant\beta_k\leqslant\beta''$, and $\|u_k\|\to\infty$ as $k\to\infty$. We may assume without loss of generality that $\lim_{k\to\infty}\beta_k=\beta\in[\beta',\beta'']$. If we set $z_k=u_k/\|u_k\|$ for $k=1,2,\ldots$, then $\|z_k\|=1$ and $z_k=A^{-1}[\alpha z_k^{-}-\beta_k z_k^{+}+\phi_1/\|u_k\|]$. Since the $L^2(\Omega)$ norm of $\alpha z_k^{-}-\beta_k z_k^{+}+\phi_1/\|u_k\|$ is bounded independently of k and A^{-1} is compact regarded as a mapping from $L^2(\Omega)$ into itself, we may assume without loss of generality that z_k converges to a function z in $L^2(\Omega)$. Clearly $\|z\|=1$ and, since $\alpha z_k^{-}-\beta_k z_k^{+}+\phi_1/\|u_k\|\to\alpha z^{-}-\beta z^{+}$in $L^2(\Omega)$ as $k\to\infty$, it follows that $z=A^{-1}[\alpha z^{-}-\beta z^{+}]$. By a bootstrap argument we infer that z is smooth and $Az+\beta z^{+}-\alpha z^{-}=0$ in Ω, $z \mid \partial\Omega=0$. If $C_1=\min(\beta-\lambda_1,\lambda_1-\alpha)>0$, then $\beta s^{+}-\alpha s^{-}-\lambda_1 s\geqslant C_1|s|$ for all s. Therefore, since $A\phi_1=-\lambda_1\phi_1$, $\phi_1 \mid \partial\Omega=0$, and $\phi_1(x)>0$ for all $x\in\Omega$, we have $0=(\phi_1, Az+\beta z^{+}-\alpha z^{-})_0=(\phi_1,\beta z^{+}-\alpha z^{-}-\lambda_1 z)_0\geqslant C_1(\phi_1,|z|)_0$. This implies that $z(x)=0$ for all $x\in\Omega$ contradicting the fact that $\|z\|=1$. This contradiction proves the lemma.

LEMMA 5.2. *Let* r *satisfy* $1<r$, $1<(\lambda_2-\lambda_1)r$, $1<(\lambda_1-\alpha)r$, $1<mr$, *and* $1<mdr$. *Let* D_k, $k=1,\ldots,4$, *be as in the statement of Proposition* 2.4 *and let* U_k, $k=1,\ldots,4$, *be as in* Lemma 2.5. *If* r_3 *is a number such that* $Au+\beta u^{+}-\alpha u^{-}=\phi_1$ in Ω, $u \mid \partial\Omega=0$, *and* $\lambda_2\leqslant\beta\leqslant\lambda_3$ *implies* $\|u\|<r_3$, $Z_k=\{u\in L^2(\Omega) \mid Pu\in U_k, \ \|(I-P)u\|\leqslant r_3\}$ *for* $k=1,\ldots,4$, *and* $K\colon L^2(\Omega)\times(\lambda_2,\infty)\to L^2(\Omega)$ *is defined by*

$$K(u,\beta)=A^{-1}[\beta u^{+}-\alpha u^{-}],$$

then $d(I+K(\cdot,\beta), Z_k, -\phi_1/\lambda_1)=(-1)^{k+1}$ for $\lambda_2<\beta<\lambda_3$ *and* $k=1,\ldots,4$.

Proof. Since m and d do not depend on β (see the proof of Lemma 2.3), r satisfies the conditions of Proposition 2.4 for all $\beta\in(\lambda_2,\lambda_3)$. It follows from Proposition 2.4, 2.5, and Lemma 2.6 that for $\beta\in(\lambda_2,\lambda_3)$ there exists a number $r_2(\beta)$ such that if $\sigma\geqslant r_2(\beta)$ and for $k=1,\ldots,4$ $Y_k(\sigma)=\{u\in L^2(\Omega) \mid Pu\in U_k, \|(I-P)u\|<\sigma\}$, then $d(I+K(\cdot,\beta), Y_k(\sigma), -\phi_1/\lambda_1)=(-1)^{k+1}$. However, according to the condition satisfied by r_3, if $\beta\in(\lambda_2,\lambda_3)$ and $u+K(u,\beta)=u+A^{-1}[\beta u^{+}-\alpha u^{-}]=A^{-1}\phi_1=-\phi_1/\lambda_1$ then $\|(I-P)u\|\leqslant\|u\|<r_3$. Hence, by the excision property of Leray–Schauder degree, if $\lambda_2<\beta<\lambda_3$ and $\sigma\geqslant r_2(\beta)$, then for $1\leqslant k\leqslant 4$ $d(I+K(\cdot,\beta), Y_k(\sigma), -\phi_1/\lambda_1)=d(I+K(\cdot,\beta), Z_k, -\phi_1/\lambda_1)=(-1)^{k+1}$.

This proves the lemma.

We shall show that $d(I+K(\cdot,\lambda_3), Z_k, -\phi_1/\lambda_2)$ is defined. For this we use two additional lemmas.

LEMMA 5.3. *If u is a solution of $Au + \lambda_3 u^+ - \alpha u \quad = \phi_1$ in Ω, $u \mid \partial\Omega = 0$, and $(Pu)(x) \geqslant 0$ on Ω, then $u^- \equiv 0$ and $u = \phi_1/(\lambda_3 - \lambda_1) + w$ where w satisfies $Pw = 0$ and*

$$Aw + \lambda_3 w = 0 \text{ in } \Omega, \qquad w \mid \partial\Omega = 0. \tag{34}$$

Proof. Let $Pu = v$ and $u = v + w$ where $w \in W$. Since $(w, \phi_1)_0 = (w, \phi_2)_0 = 0$, it follows from the variational characterization of λ_3 that

$$-(Aw, w)_0 \geqslant \lambda_3(w, w)_0 \tag{35}$$

with equality if and only if w is a solution of (34). Since $A\phi_k = -\lambda_k \phi_k$, $k = 1, 2$, and $Au + \lambda_3 u + (\lambda_3 - \alpha)\, u \quad = 0$, it follows that

$$-(Aw, w)_0 - \lambda_3(w, w)_0 - (\lambda_3 - \alpha)(u^-, w)_0 = 0. \tag{36}$$

Since $v \geqslant 0$ on Ω, $u \geqslant w$ on Ω. Therefore, $u^- w^+ \equiv 0$ on Ω and $u^- \leqslant w^-$ on Ω. Consequently, $-(u^-, w)_0 = (u^-, w^-)_0 - (u^-, w^+)_0 = (u^-, w^-)_0 \geqslant (u^-, u^-)_0$.

It follows from (36) that $-(Aw, w)_0 - \lambda_3(w, w)_0 + (\lambda_3 - \alpha)(u^-, u^-)_0 \leqslant 0$. From (35) we conclude that $u^- \equiv 0$ and $Aw + \lambda_3 w = 0$. Thus $u^+ = u$ and $Av + \lambda_3 v = A(v + w) + \lambda_3(v + w) = Au + \lambda_3 u = \phi_1$. Setting $v = s_1\phi_1 + s_2\phi_2$ we obtain $(\lambda_3 - \lambda_1)\, s_1\phi_1 + (\lambda_3 - \lambda_2)\, s_2\phi_2 = \phi_1$. Therefore $s_1 = 1/(\lambda_3 - \lambda_1)$, $s_2 = 0$, and the lemma is proved.

LEMMA 5.4. *If u is a solution of $Au + \lambda_3 u^+ - \alpha u^- = \phi_1$ in Ω, $u \mid \partial\Omega = 0$, and $(Pu)(x) \leqslant 0$ in Ω then $u = \phi_1/(\alpha - \lambda_1)$.*

Proof. Let $Pu = v$ and $u = v + w$ where $w \in W$. Since $Au + \alpha u + (\lambda_3 - \alpha)\, u^+ = 0$ we see that

$$-(Aw, w)_0 - \alpha(w, w)_0 = (\lambda_3 - \alpha)(u^+, w)_0. \tag{37}$$

Since $v \leqslant 0$ on Ω, $u \leqslant w$ on Ω and consequently, $u^+ w^- \equiv 0$ on Ω and $u^+ \leqslant w^+$ on Ω. Hence, $(u^+, w)_0 = (u^+, w^+)_0 - (u^+, w^-)_0 = (u^+, w^+)_0 \leqslant (w^+, w^+)_0$. From (37) and the inequality (35) we see that

$$(\lambda_3 - \alpha)(w, w)_0 \leqslant (\lambda_3 - \alpha)(w^+, w^+)_0. \tag{38}$$

If $w \not\equiv 0$, then since $(w, \phi_1)_0 = 0$ and $\phi_1(x) > 0$ for $x \in \Omega$, it follows that $w^+ \not\equiv 0$ and $w^- \not\equiv 0$. In particular, $w \not\equiv 0$ implies $(w^+, w^+)_0 < (w, w)_0$. As this contradicts (38) we conclude that $w \equiv 0$. Hence $u = v \leqslant 0$, so $u = -u^-$ and $u^+ = 0$. From this we see that $Av + \alpha v = \phi_1$. Setting $v = s_1\phi_1 + s_2\phi_2$ gives $(\alpha - \lambda_1)\, s_1\phi_1 + (\alpha - \lambda_2)\, s_2\phi_2 = \phi_1$. Thus, $s_1 = 1/(\alpha - \lambda_1)$, $s_2 = 0$, and the lemma is proved.

PROPOSITION 5.5. *If Z_k and K are as in Lemma 5.2, then $u+K(u, \lambda_3) \neq -\phi_1/\lambda_1$ for $u \in \partial Z_k$ and $d(I+K(\cdot, \lambda_3), Z_k, -\phi_1/\lambda_1) = (-1)^{k+1}$ for $1 \leqslant k \leqslant 4$.*

Proof. Since for $\lambda_2 < \beta < \lambda_3$ $d(I+K(\cdot, \beta), Z_k, -\phi_1/\lambda_1) = (-1)^{k+1}$, it is enough to prove that $u+K(u, \lambda_3) \neq -\phi_1/\lambda_1$ for $u \in \partial Z_k$, $1 \leqslant k \leqslant 4$, because of homotopy invariance of Leray–Schauder degree.

Suppose then that, for some k with $1 \leqslant k \leqslant 4$, $u \in \partial Z_k$ and $u+K(u, \lambda_3) = u + A^{-1}[\lambda_3 u^+ - \alpha u^-] = -\phi_1/\lambda_1$. By a bootstrap argument, u is smooth, $Au + \lambda_3 u^+ - \alpha u^- = \phi_1$ in Ω, and $u \mid \partial\Omega = 0$. According to the way r_3 was chosen (see Lemma 2.5), it follows that $\|(I-P)u\| \leqslant \|u\| < r_3$. Hence, $Pu \in \partial U_k$. If $k=1$, then according to Lemma 2.2, $v \equiv Pu \geqslant 0$ on Ω. Therefore, by Lemma 5.3, $u = \phi_1/(\lambda_3 - \lambda_1) + w$ where $Aw + \lambda_3 w = 0$, $w \mid \partial\Omega = 0$. Since $\lambda_2 < \lambda_3$, $(w, \phi_1)_0 = (w, \phi_2)_0 = 0$ so $Pw = 0$ and $Pu = \phi_1/(\lambda_3 - \lambda_1)$.

Since $1/(\lambda_3 - \lambda_1) < 1/(\lambda_2 - \lambda_1) < r$, $(1/(\lambda_3 - \lambda_1), 0)$ is an interior point of D_1 which means that Pu is an interior point of U_1. This contradiction shows that $Pu \notin \partial U_1$.

If $Pu \in U_3$, then, according to Lemma 2.2, $Pu \leqslant 0$ on Ω. It follows from Lemma 5.4 that $u = \phi_1/(\alpha - \lambda_1)$ so $Pu = u$. Since $-r < 1/(\alpha - \lambda_1)$, the point $(1/(\alpha - \lambda_1), 0)$ is interior to D_3 which means that Pu is interior to U_3. This contradiction shows that $Pu \notin \partial U_3$.

If $Pu \in \partial U_2$, then since $Pu \notin \partial U_3$ we must have $Pu = s_1\phi_1 + s_2\phi_2$ with $|s_1| \leqslant r$ and $s_2 = mr$. Referring to the proof of Lemma 2.3 we have $1 = (\phi_1, \phi_1)_0 = (\phi_1, Au + \lambda_3 u^+ - \alpha u^-)_0 = (\lambda_3 u^+ - \alpha u^- - \lambda_1 u, \phi_1)_0 = ((\lambda_3 - \lambda_1)u^+ + (\lambda_1 - \alpha)u^-, \phi_1)_0 \geqslant C(|u|, \phi_1)_0 \geqslant Cm\,(|u|, |\phi_2|)_0 \geqslant Cm|(u, \phi_2)_0| = ds_2 = dmr$. Since $dmr > 1$, we have a contradiction. Hence $Pu \in \partial U_2$ is impossible.

Finally, suppose that $Pu \in \partial U_4$. Since we have shown that it is impossible that $Pu \in \partial U_1$ or $Pu \in \partial U_3$, it follows that $Pu = s_1\phi_1 + s_2\phi_2$ where $|s_1| \leqslant r$ and $s_2 = -mr$. The exact same reasoning used in the previous paragraph gives a contradiction. Hence $Pu \notin \partial U_4$ and the first assertion of the proposition is proved. By earlier remarks the second assertion is also proved.

THEOREM 5.1. *Let f be as in (2) with $\beta = \lambda_3$ and let g and h be as in Section 2. There exists $s_1 > 0$ such that (11) has at least four solutions for $s > s_1$.*

Proof. This theorem follows from Proposition 5.5 in exactly the same way that Theorem 3.1 follows from Lemma 2.6 and Lemma 3.1.

From the definition of K given in Lemma 5.2 we have $K(u, \beta) - K(u, \lambda_3) = A^{-1}(\beta - \lambda_3)u^+$. Since for $k = 1, \ldots, 4$ $\inf_{u \in \partial Z_k} \|u + K(u, \lambda_3) + \phi_1/\lambda_1\| > 0$ we infer the existence of a number $\varepsilon > 0$ such that for $k = 1, \ldots, 4$ $\lambda_3 < \beta \leqslant \lambda_3 + \varepsilon$ and $u \in \partial Z_k$, $\|K(u, \beta) - K(u, \lambda_3)\| < \|u + K(u, \lambda_3) + \phi_1/\lambda_1\|$. Therefore, by the Poincare–Bohl theorem we have

PROPOSITION 5.6. *If* $\lambda_3 \leqslant \beta \leqslant \lambda_3 + \varepsilon$ *then for* $1 \leqslant k \leqslant 4$, $\|u + K(u, \beta) + \phi_1/\lambda_1\| \neq 0$ *for* $u \in \partial Z_k$ *and* $d(I + K(\ , \beta), Z_k, -\phi_1/\lambda_1) = (-1)^{k+1}$.

In the remainder of this section we assume that the interval $(\lambda_3, \lambda_3 + \varepsilon]$ does not contain an eigenvalue of the problem $A\phi + \lambda\phi = 0$, $\phi \mid \partial\Omega = 0$. This can be arranged by taking $\varepsilon > 0$ smaller if necessary.

The main result of this section is

THEOREM 5.2. *Let* λ_3 *be an eigenvalue of odd multiplicity. If* $\lambda_3 < \beta \leqslant \lambda_3 + \varepsilon$, *then for* s *sufficiently large,* (11) *has at least five solutions.*

Proof. We fix $\beta \in (\lambda_3, \lambda_3 + \varepsilon]$ and denote by z the unique solution of $Au + \beta u = \phi_1$, $u \mid \partial\Omega = 0$ given explicitly by $z = \phi_1/(\beta - \lambda_1)$. The proof of Theorem 5.2 will be based largely on

LEMMA 5.7. *Let* β *be as in Theorem* 5.2. *There exists* $\sigma > 0$ *such that if* $u \varepsilon L^2(\Omega)$ *and satisfies* $\|u - z\| \leqslant \sigma$ *and*

$$u + A^{-1}[\beta u^+ - (\alpha + s(\beta - \alpha))\, u^-] + \phi_1/\lambda_1 = 0 \tag{39}$$

where $0 \leqslant s \leqslant 1$, *then* $u = z$.

Proof. We define a finite increasing sequence of numbers $\{p_k\}_1^m$ with $m \geqslant 1$ as follows: let $p_1 = 2$. If $n \equiv \dim \Omega = 1$, set $m = 1$. Otherwise, assume that p_j has been defined for $1 \leqslant j \leqslant k$, where $k \geqslant 1$, $p_k \leqslant n$, and $p_j = np_{j-1}/(n - 2p_{j-1})$ for $2 \leqslant j \leqslant k$ if $k \geqslant 2$. If $2p_k \geqslant n$ set $p_{k+1} = q$ where q is a fixed number with $q > n$ and set $m = k + 1$. Otherwise, set $p_{k+1} = np_k/(n - 2p_k)$. If $p_{k+1} > n$ set $m = k + 1$; otherwise define p_{k+2} in terms of p_{k+1} as above. Eventually this process must come to a stop, for in the contrary case $\{p_k\}_1^\infty$ would be an increasing sequence with $p_k \leqslant n$ for all k and $p^* \equiv \lim_{k \to \infty} p_k$ would have to satisfy $p^* = n\, p^*/(n - 2p^*)$, which is absurd.

Let $\|\ \|_r$ denote the norm in $L^r(\Omega)$ for $1 \leqslant r < \infty$. Since $p_m > n$, it follows from the Sobolev embedding theorem that $W_{2,p_m}(\Omega)$ is continuously embedded in $C^1(\bar{\Omega})$. Since according to standard elliptic theory, $A^{-1}\colon C^\gamma(\bar{\Omega}) \to \dot{C}^{2+\gamma}(\bar{\Omega})$ extends to a continuous linear map from $L^{p_m}(\Omega)$ onto $W_{2,p_m}(\Omega)$, there exists a constant $d_m > 0$ such that

$$\|A^{-1}u_1 - A^{-1}u_2\|_{C^1(\bar{\Omega})} \leqslant d_m\, \|u_1 - u_2\|_{p_m} \qquad \text{for all } u_1, u_2 \in L^{p_m}(\Omega). \tag{40}$$

Consider the case $m \geqslant 2$ and let $1 \leqslant j \leqslant m - 1$. Since A^{-1} extends as a continuous linear map from $L^{p_j}(\Omega)$ onto $W_{2,p_j}(\Omega)$, and since $W_{2,p_j}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for $1 \leqslant r \leqslant np_j/(n - 2p_j)$ if $2p_j < n$, and in $L^r(\Omega)$ for $1 \leqslant r < \infty$ for $2p_j \geqslant n$, it follows from the definition of p_{j+1} that there exists $d_j > 0$ such that for any $u_1, u_2 \in L^{p_j}(\Omega)$

$$\|A^{-1}u_1 - A^{-1}u_2\|_{p_{j+1}} \leqslant d_j\, \|u_1 - u_2\|_{p_j}. \tag{41}$$

Suppose that for some s in $[0, 1]$ $u \in L^2(\Omega)$ and u satisfies (39). By a bootstrap argument, u is smooth. Moreover, since $z + A^{-1}\beta z + \phi_1/\lambda_1 = 0$ we have

$$u - z = -A^{-1}[\beta u^+ - (\alpha + s(\beta - \alpha))\, u^- - \beta z], \tag{42}$$

Let $\omega = \max(\beta, |\alpha|)$. Since $z = z^+$ and $z^- = 0$, it follows that for $x \in \Omega$, $|\beta u^+(x) - (\alpha + s(\beta - \alpha))\ u^-(x) - \beta z(x)| \leqslant \beta\, |u^+(x) - z^+(x)| + |\alpha + s(\beta - \alpha)|\ |u^-(x) - z^-(x)| \leqslant (\beta + |\alpha + s(\beta - \alpha)|)\, |u(x) - z(x)|$. Hence, for $x \in \Omega$

$$|\beta u^+(x) - (\alpha + s(\beta - \alpha)\, u^-(x) - \beta z(x)| \leqslant 2\omega\, |u(x) - z(x)|. \tag{43}$$

Combining (40), (42), and (43) we obtain

$$\|u - z\|_{C^1(\bar{\Omega})} \leqslant 2\omega d_m\, \|u - z\|_{p_m}. \tag{44}$$

If $m \geqslant 2$ and $1 \leqslant j \leqslant m - 1$, then from (41)–(43) it follows that

$$\|u - z\|_{p_{j+1}} \leqslant 2\omega d_j\, \|u - z\|_{p_j}. \tag{45}$$

From (44) and (45) we see that

$$\|u - z\|_{C^1(\bar{\Omega})} \leqslant \bar{\omega}\, \|u - z\|_{p_1} = \bar{\omega}\, \|u - z\| \tag{46}$$

where $\bar{\omega} = 2\omega d_1$ if $m = 1$ and $\bar{\omega} = (2\omega)^m\, d_1 \cdots d_m$ if $m > 1$.

Since $\phi_1 > 0$ on Ω and $\partial\phi_1/\partial n < 0$ on $\partial\Omega$ there exists a number $\eta > 0$ such that if $v \in C^1(\bar{\Omega})$, $v \mid \partial\Omega = 0$, and $\|v - z\|_{C^1(\bar{\Omega})} \leqslant \eta$ then $v(x) \geqslant 0$ for all $x \in \Omega$. Let $\sigma = \eta/\bar{\omega}$. If $u \in L^2(\Omega)$, $\|u - z\| \leqslant \sigma$, and u satisfies (39) for some $s \in [0, 1]$, then by (46) $\|u - z\|_{C^1(\bar{\Omega})} \leqslant \eta$ so $u(x) \geqslant 0$ on Ω. From (39) we see that $u + A^{-1}(\beta u) + \phi_1/\lambda_1 = 0$. Hence, $Au + \beta u = \phi_1$, $u \mid \partial\Omega = 0$. Since β is not an eigenvalue there is only one solution of this problem. Hence $u = z$ and the lemma is proved.

Let $D_\tau = \{u \in L^2(\Omega)\ \|u - z\| < \tau\}$. If $0 < \tau < \sigma$, then according to the lemma $u + A^{-1}[\beta u^+ - (\alpha + s(\beta - \alpha))\, u^+] + \phi_1/\lambda_1 \neq 0$ for $u \in \partial D_\tau$ and $0 \leqslant s \leqslant 1$. By considering the cases $s = 0$ and $s = 1$ and using the homotopy invariance property of degree, we conclude that if $0 < \tau \leqslant \sigma$, then

$$d(I + K(\cdot, \beta), D_\tau, -\phi_1/\lambda_1) = d(I + \beta A^{-1}, D_\tau, -\phi_1/\lambda_1). \tag{47}$$

Since $u + \beta A^{-1}u + \phi_1/\lambda_1 = 0$ has the unique solution $u = z$, it follows from a standard result of degree theory that $d(I + \beta A^{-1}, D_\tau, -\phi_1/\lambda_1) = (-1)^r$ where r is the number of eigenvalues μ, with $\mu > 1$, counting multiplicities, of the linear problem $\mu\phi + \beta A^{-1}\phi = 0$ (see [13, p. 66]). If ϕ is a nonzero solution of $\mu\phi + \beta A^{-1}\phi = 0$ with $\mu > 1$ then $A\phi + (\beta/\mu)\,\phi = 0$, $\phi \mid \partial\Omega = 0$, so $\mu = \beta/\lambda_k$ for some $k \geqslant 1$. Since the only possible solutions with $\mu > 1$ are

$\mu=\beta/\lambda_1$, $\mu=\beta/\lambda_2$, and $\mu=\beta/\lambda_3$, and since λ_1 and λ_2 are simple and λ_3 has odd multiplicity, we conclude that r is odd. Thus, by (47), it follows that for $0<\tau\leqslant\sigma$,

$$d(I+K(\cdot,\beta), D_\tau, -\phi_1/\lambda_1) = -1. \tag{48}$$

Since z is an interior point of Z_1 (see Lemma 5.2) we may choose a number σ_0 with $0<\sigma_0\leqslant\sigma$ such that the closed ball $\bar{D}_{\sigma_0}$ is in the interior of Z_1. By that additivity property of degree, Lemma 5.2, and (48) we see that

$$d(I+K(\cdot,\beta), Z_1-\bar{D}_{\sigma_0}, -\phi_1/\lambda_1)=2. \tag{49}$$

The proof of Theorem 5.2 now follows by the same reasoning used to prove Theorem 5.1: Let $T_s: L^2(\Omega)\to L^2(\Omega)$ be defined by $T_s(\psi)= A^{-1}[\beta\psi^+ - \alpha\psi^- + f_0(s\psi)/s - h/s] = K(\psi,\beta) + A^{-1}[f_0(s\psi)/s - h/s]$. Because of condition (10), if s is large then $\|T_s(\psi)-K(\psi,\beta)\| < \|\psi+K(\psi,\beta)+\phi_1/\lambda_1\|$ for all $\psi\in\partial Z_k$ for $k=1,\ldots,4$ and for all $\psi\in\partial D_{\sigma_0}$. Consequently, for $s>0$ sufficiently large, it follows from Proposition 5.6, (48), and (49) that $d(I+T_s, Z_k, -\phi_1/\lambda_1)=(-1)^{k+1}$ for $k=2,3,4$, $d(I+T_s, D_{\sigma_0}, -\phi_1/\lambda_1)=-1$, and $d(I+T_s, Z_1-\bar{D}_{\sigma_0}, -\phi_1/\lambda_1)=2$ for s sufficiently large. Thus, for such values of s, there exist functions ψ_j, $j=1,\ldots,5$, such that $\psi_j+T_s(\psi_j)+\phi_1/\lambda_1=0$ with $\psi_j\in Z_j$ for $j=2,3,4$, $\psi_1\in Z_1-\bar{D}_{\sigma_0}$, and $\psi_5\in D_{\sigma_0}$. If, for such values of s, we set $u_j=s\psi_j$ for $j=1,\ldots,5$ then u_j is a solution of (11) and the proof of Theorem 5.2 is complete.

6. Results on the Piecewise Linear Problem for a Square Region

In this section we let Ω denote the square region in R^2 given by $0<x<\pi$ and $0<y<\pi$. We consider the boundary value problem

$$\Delta u+\beta u^+ - \alpha u^- = \sin x \sin y \text{ in } \Omega, \qquad u \mid \partial\Omega=0 \tag{50}$$

where Δ is the Laplacian operator. For the eigenvalue problem $\Delta u+\lambda u=0$ in Ω, $u\mid\partial\Omega=0$ it is known—see, for example, [6, chap. 11]—that $\lambda_1=2$, $\lambda_2=\lambda_3=5$, $\lambda_4=8$, and $\lambda_5=\lambda_6=10$, with corresponding eigenfunctions $\phi_1(x,y)=\sin x\sin y$, $\phi_2(x,y)=\sin 2x\sin y$, $\phi_3(x,y)=\sin x\sin 2y$, $\phi_4(x,y)=\sin 2x\sin 2y$, $\phi_5(x,y)=\sin x\sin 3y$, and $\phi_6(x,y)=\sin 3x\sin y$. As an application of the results and methods of the previous sections we sketch the proof of the following result which supports the conjecture discussed in the introductory section:

Theorem 6.1. *Let $\alpha<\lambda_1$. If $\lambda_3<\beta<\lambda_4$, then (50) has at least six dis-*

tinct solutions and, if $\lambda_4 < \beta < \lambda_5$, then (50) *has at least eight distinct solutions.*

Proof. We consider first the case $\alpha < \lambda_1 = 2$ and $5 = \lambda_3 < \beta < \lambda_4$. We look for all solutions of (50) which have the special form

$$u(x, y) = \theta(y) \sin x. \tag{51}$$

Since the nonlinearity in (50) is positively homogeneous of degree one this leads to the conditions

$$\theta''(y) - \theta(y) + \beta\theta^+(y) - \alpha\theta^-(y) = \sin y$$

or

$$\theta''(y) + (\beta - 1)\,\theta^+(y) - (\alpha - 1)\,\theta^-(y) = \sin y \tag{52}$$

and

$$\theta(0) = \theta(\pi) = 1. \tag{53}$$

Since $\alpha - 1 < 1$ and $4 < \beta - 1 < 7$ and since the first three eigenvalues of the linear problem $Z'' + \mu Z = 0$ $Z(0) = Z(\pi) = 0$ are $\mu_1 = 1$, $\mu_2 = 4$, and $\mu_3 = 9$, it follows from the remark following the statement of Proposition 2.4 that the problem (52), (53) has at least four distinct solutions. By symmetry there are at least four solutions of (52), (53) which have the form

$$u(x, y) = \theta(x) \sin y. \tag{54}$$

A solution u of (50) will have both of the form (51) and (52) if and only if $u(x, y) = c \sin x \sin y$. A solution of (50) with this form will be a solution of

$$\Delta u + \beta u = \sin x \sin y, \qquad u \mid \partial\Omega = 0 \tag{55}$$

if $c > 0$, and a solution of

$$\Delta u + \alpha u = \sin x \sin y, \qquad u \mid \partial\Omega = 0 \tag{56}$$

if $c < 0$. Since (55) has the unique solution $u(x, y) = \sin x \sin y/(\beta - 2)$ and (56) has the unique solution $u(x, y) = \sin x \sin y/(\alpha - 2)$, there are exactly two solutions of (50) which have both the forms (51) and (54). Thus, the conditions $\alpha < \lambda_1$ and $\lambda_3 < \beta < \lambda_4$ imply the existence of at least six distinct solutions of the boundary value problem (50).

Next we consider the case where $\alpha < \lambda_1$ and $\lambda_4 < \beta < \lambda_5$. Since $\alpha - 1 < 1$ and $7 < \beta - 1 < 9$, the boundary value problem (52), (53) again has at least four distinct solutions. Therefore, there are at least four solutions of (50) which have the form (51) and at least four solutions which have the form (54). As before, there are exactly two solutions of (50) which have both these forms so we have found at least six distinct solutions.

To find additional solutions we look for solutions of (50) which satisfy the condition

$$u(x, y) = u(y, x). \tag{57}$$

Let $S\colon C(\bar{\Omega}) \to C(\bar{\Omega})$ be defined by $(Su)(x, y) = u(y, x)$. If $u \in C^2(\bar{\Omega})$ it is easy to see that $\Delta Su = S\Delta u$. If $L^2_s(\Omega)$ denotes the set of elements in $L^2(\Omega)$ which satisfy (56) almost everywhere and $\mathring{H}^s_2(\Omega)$ denotes the set of elements in $H_2(\Omega) \cap \mathring{H}_1(\Omega)$ which satisfy (57) then Δ extends to a continuous linear map Δ_s from $\mathring{H}^s_2(\Omega)$ onto $L^2_s(\Omega)$ which is injective. (This follows from standard elliptic theory.) The inverse of this map composed with the injection of $\mathring{H}^s_2$ into $L^2_s(\Omega)$ is compact. If $\phi \in \mathring{H}^s_2(\Omega)$, $\phi \equiv 0$, and $\Delta\phi + \lambda\phi = 0$ then ϕ is of class C^2 and satisfies the boundary condition $u \mid \partial\Omega = 0$. Since it is known (see [6, chap. 11]) that functions of the form $\sin px \sin qx$ with $p = 1, 2,..., q = 1, 2,...,$ form a complete set of eigenfunctions of the Laplacian for the square with Dirichlet boundary condition; the eigenvalues of Δ_s are 2, 8, 18, 32,.... These are all simple with corresponding eigenfunctions $\sin px \cos py$, $p = 1, 2, 3....$

Letting $\phi_1(x, y) = \sin x \sin y$ and $\phi_2(x, y) = \sin 2x \sin 2y$ we see that for constants C_1 and C_2 we have $C_1\phi_1(x, y) + C_2\phi_2(x, y) = (C_1 + 4C_2 \cos x \cos y) \sin x \sin y$. Therefore if $|C_2| \leqslant |C_1|/4$, $C_1\phi_1 + C_2\phi_2$ does not change sign on Ω and its sign is the same as that of C_1 if $C_1 \neq 0$. Since $\alpha < 2$ and $8 < \beta < 10$, with the reasoning of the second section with A replaced by Δ_s and $L^2(\Omega)$ replaced by $L^2_s(\Omega)$ we infer the existence of at least four distinct solutions of $\Delta^s_s u + \beta u^+ - \alpha u^- = \sin x \sin y$ in $\mathring{H}^s_2(\Omega)$. Since u^+ and u^- are in $\mathring{H}_1(\Omega)$ (see [16]) and Ω satisfies the cone condition, it follows that the solutions are in $H_3(\Omega)$. Hence by the Sobolev embedding theorem these solutions are in $C^1(\bar{\Omega})$ and satisfy the boundary condition $u \mid \partial\Omega = 0$. Interior Schauder estimates show that these solutions are smooth inside Ω. Hence, there are at least four classical solutions of (50) which satisfy the condition (57).

A solution of (50) which satisfies condition (57) and has either of the forms (51) or (54) must necessarily satisfy $u(x, y) = c \sin x \sin y$. As shown in the proof of the first part of the theorem, there are exactly two solutions of this type. Therefore, from the four solutions satisfying (57) we get two solutions distinct from the six already found. This proves the result.

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